I want to be careful that I don’t leave you with the impression that the Shapley Value is the only fair way to divide the pie. I think it has many desirable properties, but it isn’t the only option. Here I’ll explain one other approach, called the Nucleolus, an idea developed by David Schmeidler in 1969.

To explain how the Nucleolus works, let’s take a slightly different version of the runway problem. We have three airlines.

* A needs a runway of length 12
* B needs a runway of length 18
* C needs a runway of length 18

Under the Shapley Value approach, all three airlines would split the cost of the first length three ways and then B and C, as the only users of the next half length, would share that cost equally. Thus

A pays 4

B pays 4 + 3 = 7

C pays 4 + 3 = 7

Before turning to the Nucleolus, we can state one property any fair solution should obey. Whatever amount A pays, it should no more than what B or C pays. This follows as A is using less of the runway than B or C.

From this it also follows that B and C should pay the exact same amount, as B uses no more than C and C uses no more than B. What this means is that the most A should pay in this situation is 6. The reason is that since B and C must pay at least as much as A, once A pays 6 so must B and C and together that would cover the full cost of the runway.

At the other extreme, the least A should pay is 4. It wouldn’t be fair for A to pay less than 4 as that it its equal share of the runway it uses. And it wouldn’t be fair for A to pay more than 6 as that would mean that it pays more than the other two airlines which use more of the runway. Thus, the full range of what should be under consideration is A pays something in the 4 to 6 range.

The Nucleolus makes the argument for why A should pay 6. The motivation comes from looking at how much each side gains when one individual (or group) joins another. We can illustrate this perspective using the Shapley Value solution.

Imagine that B and C have already formed a partnership. In that case, the two of them have already saved 18 by coming together. If A joins them and makes it a three-way partnership, that will create another 12 of savings. Under the Shapley Value, A only pays 4 and thus saves 8 while the (B, C) partnership only saves 4. Thus, Airline A gains double what the (B, C) partnership gets. This seems unfair. A needs the (B,C) groups just as much as the (B, C) group needs A. The Nucleolus proposes that this gain be split evenly, namely 6 and 6, which requires A to pay 6.

You might be wondering why I picked the combination of A joining (B, C). Why not look at B joining an (A,C) partnership? Indeed, the Nucleolus looks at all these possible combinations. When B joins (A, C) there is a gain of 18. Using the division of the Shapley Value, B would pay 7 and thus gain 11, while (A, C) would gain 7. This also isn’t equal. To make this equal, we’d have to have B pay 9 and then by symmetry C would have to pay 9, but that means A would pay nothing. In that case, we’ve made the asymmetry even worse in the first case, the one in which A joins (B, C). If A pays nothing, then all of the gain goes to A and none to (B, C).

What the Nucleolus does is find the division that maximizes the smallest gain. And once that it done, it maximizes the next smallest gain subject to not lowering the smallest gain. It isn’t possible to make the all the gains equal and so the Nucleolus comes as close as possible.

Recall that when A joins (B, C) there is only 12 to go around. If we split this evenly, A gains 6 which implies it pays 6. And since B and C must not pay less than A, they pay 6 as well. We know that the gain from A joining (B, C) are split evenly. Let’s check the result for when B joins (A, C). Since B pays 6, too, it gains 12 while (A, C) gains 6. And the same result is true for when C joins (A, B). We might like to increase how much (A, C) gains as the division is lopsided toward B. But, to do so would require that B pays more and, by a symmetric argument, that C pays more. But if B and C each pay strictly more than 6, then the (B, C) pair will pay strictly more than 12 and that means they will gain less than 6 when A joins them. Thus we can’t make the result for when B joins (A, C) any more fair without making the A joins (B, C) even less fair.

John Rawls argued that society should work to maximize the welfare of its worst-off members. In a similar vein, the Nucleolus works to find the cost division that maximizes the gain to the group that is getting the least from coming together. And it keeps on doing that to the extent possible.

Let me provide three more illustrations.

* A needs a runway of length 12
* B needs a runway of length 24
* C needs a runway of length 24

Under the Shapley Value approach, we’d say that all three airlines would split the cost of the first length three ways and then B and C, as the only users of the next length, would share that cost equally. Thus

A pays 4

B pays 4 + 6 = 10

C pays 4 + 6 = 10

But note here that under this cost division when A joins (B, C), A gains 8 and (B, C) only gains 4. To equalize this, we should have A pay 6. Thus the Nucleolus solution is A pays 6 B pays 9 C pays 9 Unlike our first example, it is not the case here that all three parties split the cost evenly. Indeed, as the runway need of Airlines B and C increases, they pay all of the additional costs. A never pays more than half the cost of the first length.

And to the extent that the runway needs of B and C shrink, then the three airlines continue splitting the full costs three ways. For example, with the numbers below, A, B, and C would each pay 5 in the Nucleolus.

* A needs a runway of length 12
* B needs a runway of length 15
* C needs a runway of length 15

You might still be wondering why it is fair for A to be paying an equal share of the full runway cost. The reason is that A is very lucky to be joining (B, C). The two of them can already create a large amount of savings without A. Thus A should be happy to split the surplus that is created when it joins this pair.

This becomes even clearer when we add additional airlines that need the longer runway. Consider the case:

* A needs a runway of length 12
* B needs a runway of length 16
* C needs a runway of length 16
* D needs a runway of length 16

Under the Shapley Value A pays 3 while under the Nucleolus A pays 4. Either way, A gains a great amount from joining the (B, C, D) trio. It is either 9 or 8 ahead while the trio gains 3 or 4. To make the gains as equal as possible, A should pay 4. A can never pay more than 4 as that would mean it pays more than B, C, and D.

I don’t want to pretend that this is a rigorous explanation of the Nucleolus. For more information have a look at the original article by Schmeidler and an elegant application of the Nucleolus to our airport cost-sharing problem by Littlechild.

Littlechild, S.C. (1974) A simple expression for the nucleolus in a special case. Int J Game Theory 3:21–29

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