A New Approach to the Shapley Value

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Preview of results

New justification and more general procedure that leads to Shapley value

Under the Shapley procedure, player joining coalition keeps entire marginal contribution

Under our procedure, gains are divided up $(\alpha, 1 - \alpha)$

Result is still the Shapley value for all α

Surprising since payoffs must depend *only* on marginal contributions for Shapley value

Preview of results

New non-cooperative game that mimics our procedure (and leads to Shapley value)

- Builds on Hart and Mas-Colell (1996)
  - But can get Shapley value without having person making offer be one subject to elimination
- Leads to a different way of thinking about weights in a Shapley value
  - Endogenous weights as opposed to fixed weights

Results can be extended to NTU games

Standard Procedure

Players join existing coalitions in random order

Player joining gets full marginal contribution

Shapley value is expected value of marginal contribution over all possible orderings

$$\phi_i(v) = \frac{1}{n!} \sum_R \left[v\left(P_i^R \cup \{i\}\right) - v\left(P_i^R\right) \right]$$

Why would players in a real-world bargaining game ever agree to a Shapley value outcome?

Shapley himself addressed this second question at the conclusion of his original paper. As is well-known, he introduced the following probabilistic model of a constructive nature that yielded his value: each player is awarded his expected marginal contribution to the set of players that precede him, where the expectation is taken with respect to the uniform distribution over the set of all orders of the players. A criticism of this scheme is that a player is awarded his *entire* (utility) contribution to a coalition. This violates an intuitive symmetry whereby we would expect the player to receive only a portion of his contribution to the coalition (but see Rothblum [29] for an alternative interpretation).

Horace W. Brock (1992)

New Procedure

Players join existing coalitions in random order

Player joining gets α and the others get $1 - \alpha$ of the marginal contribution

More realistic for person joining the coalition not to have all the bargaining power

If $\alpha = 1$ then identical to Shapley procedure

If $\alpha < 1$ must decide how the other players split up the $1 - \alpha$

Claims are based on marginal contributions

Why We Care about Procedures

Not a set of axioms. Not a non-cooperative game.

In-between a cooperative game with axioms and a non-cooperative game rules.

A guide for the players' actions.

May be persuasive tool to get participants to agree division of value.

Helps us understand the interplay between the axioms.

We don't stop at procedure. Provide non-cooperative game as well.

How do we split up the $1 - \alpha$?

Split $1 - \alpha$ share evenly doesn't work well

V(A) = V(B) = V(C) = 0. V(AB) = V(AC) = 0. V(BC) = 1, V(ABC) = 1.

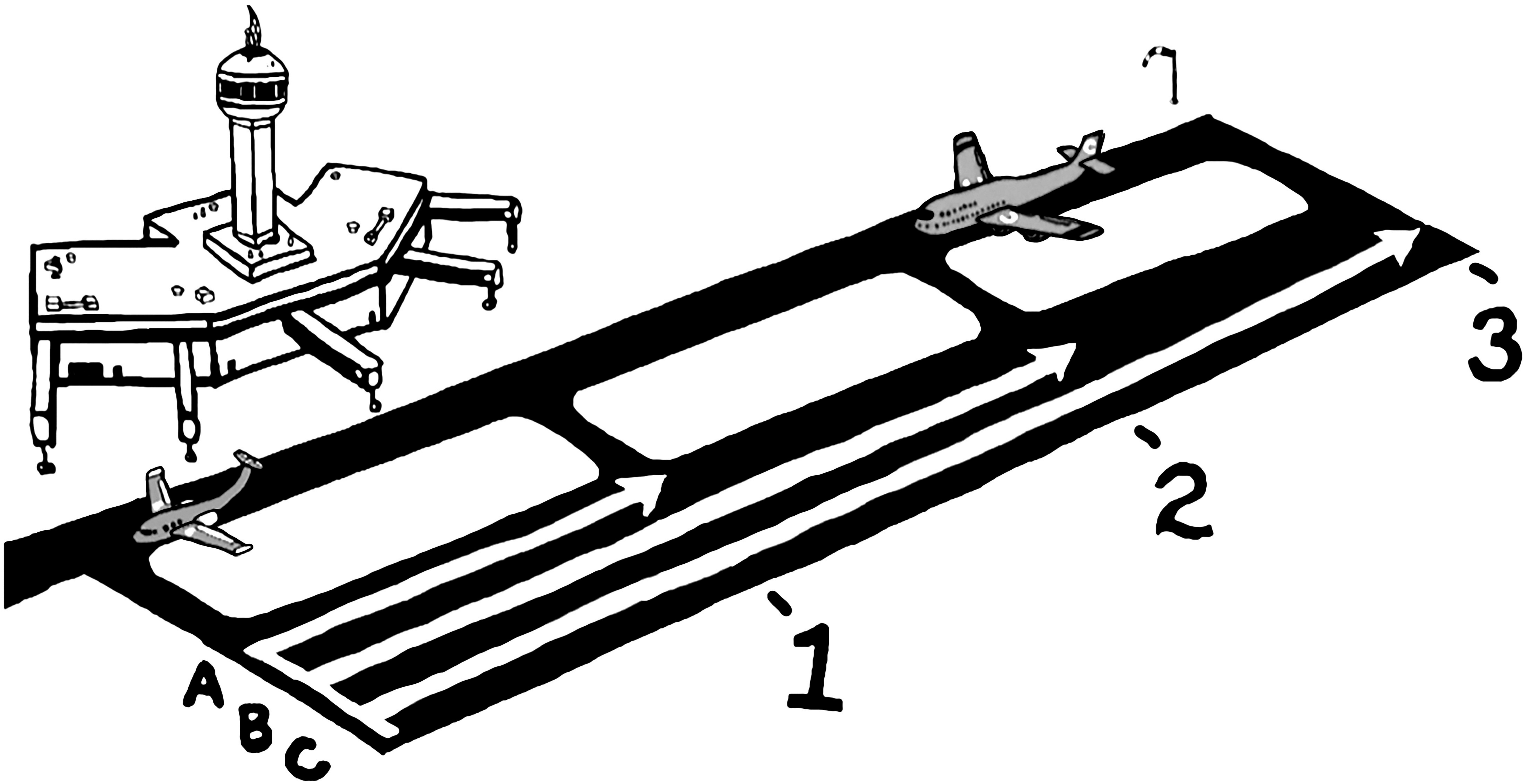
A is a dummy player. Outcome should be B and C each get 0.5.

A joins {BC}: A gets an α share of 0
B joins {AC}: A gets 1/2 $(1 - \alpha)$
C joins {AB}: A gets 1/2 $(1 - \alpha)$

On average, A gets 1/3 [0 + 1/2 $(1 - \alpha)$ + 1/2 $(1 - \alpha)$] = $(1 - \alpha)$/3.

Same as what B and C get from their $1 - \alpha$ shares. Payoffs don't depend on marginal contributions. Also, violates dummy player axiom.

Shares should depend on marginal contributions.



The Airport Game. Littlechild and Owen (1973)

Why share only up to your marginal contribution

Cost(A)=1; Cost (B)=2; Cost(C)=3.   Cost (AB)=2, Cost (AC)=3, Cost(BC)=3   Cost(ABC)=3

$m_A = 1,\ m_B = 2,\ m_C = 2$ (the marginal contributions to ABC)

A joins {BC}   A saves α, B and C equally split the $(1-\alpha)$

Why share only up to your marginal contribution

Cost(A)=1; Cost (B)=2; Cost(C)=3. Cost (AB)=2, Cost (AC)=3, Cost(BC)=3 Cost(ABC)=3

$m_A = 1, m_B = 2, m_C = 2$

A joins {BC} A saves α, B and C equally split the $(1-\alpha)$

B joins {AC} B saves $\alpha 2$, A and C equally split the $(1-\alpha)$ associated with first leg

C gets all $(1-\alpha)$ savings on 2nd

A didn't pay anything for second leg.

Shouldn't share in the savings.

C joins {AB} Similar to B joins {AC}

Motivating example

$$V(A) = V(B) = V(C) = 0$$

$$V(AB) = 20;\ \ V(AC) = 18;\ \ V(BC) = 12$$

$$V(ABC) = 24$$

Henceforth, we will ignore V(A) = V(B) = V(C) = 0

Note that core is empty in this example

Motivating example: A joins B or B joins A

$V(AB) = 20$ implies A and B each have marginal contribution of 20

With two players, no issue how to split up the $1 - \alpha$ since there is only one person

Result doesn't depend on α. For any α, they split the value created.

Across AB and BA, expected payoffs are

When A joins B A gets $\alpha 20$

When B joins A A gets $(1 - \alpha)20$

On average, A gets ½ [$\alpha 20$ + $(1 - \alpha)20$] = 10

Motivating example: C joins {AB}

$$V(AB) = 20;\ \ V(AC) = 18;\ \ V(BC) = 12. \quad\quad V(ABC) = 24$$

Marginal contributions: $m_A = 12$, $m_B = 6$, $m_C = 4$

C gets $\alpha 4$ and {AB} each get $10\ +\ \frac{1}{2}(1-\alpha)4$

A and B split the $(1-\alpha)4$ equally since each has a claim on the full amount

Motivating example: B joins {AC}

$$V(AB) = 20;\; V(AC) = 18;\; V(BC) = 12. \quad V(ABC) = 24$$

Marginal contributions: $m_A = 12$, $m_B = 6$, $m_C = 4$

B gets $\alpha 6$ and {AC} each get 9 but they do not split the $(1 - \alpha)6$ equally

A and C split $(1 - \alpha)4$ equally and A gets remaining $(1 - \alpha)2$ all to itself

C's claim is limited to 4

Motivating example: A joins {BC}

$$V(AB) = 20; \; V(AC) = 18; \; V(BC) = 12. \quad V(ABC) = 24$$

Marginal contributions: $m_A = 12$, $m_B = 6$, $m_C = 4$

A gets $\alpha 12$ and {BC} each get 6 but they do not split the $(1-\alpha)12$ equally

B and C split $(1-\alpha)4$ equally and B gets the remaining $(1-\alpha)2$ all to itself

That leaves $(1-\alpha)6$ unclaimed, so it goes back to A!

Looks messy. Everything depends on α

What does C get?

C joins {AB}	$\alpha 4$
B joins {AC}	$9 + \frac{1}{2}(1 - \alpha)4$
A joins {BC}	$6 + \frac{1}{2}(1 - \alpha)4$
On average	$\frac{1}{3}\left[15 \ + \alpha 4 + (1 - \alpha)4\right] = \frac{19}{3}$

Gets full $\alpha 4$ when on the outside and half its $(1 - \alpha)4$ twice when on the inside

Payoff is independent of α. Hence same as $\alpha = 1$. Hence Shapley value!

What does B get?

C joins {AB} $\quad 10 + \frac{1}{2}(1-\alpha)4$

B joins {AC} $\quad \alpha 6$

A joins {BC} $\quad 6 + \frac{1}{2}(1-\alpha)4 + (1-\alpha)2$

On average $\quad \dfrac{1}{3}[16 + \alpha 6 + (1-\alpha)6] \;=\; 22/3$

Gets full $\alpha 6$ when on the outside and half its $(1-\alpha)4$ twice when on the inside similar to C plus extra $(1-\alpha)2$ when paired with C for total of $(1-\alpha)6$.

What does A get?

C joins {AB} $$10 + \frac{1}{2}(1-\alpha)4$$

B joins {AC} $$9 + \frac{1}{2}(1-\alpha)4 + (1-\alpha)2$$

A joins {BC} $$\alpha 12 + (1-\alpha)6$$

On average $$\frac{1}{3}\left[19 \ + \alpha 12 + (1-\alpha)12\right] = 31/3$$

Gets full $\alpha 12$ when on the outside and half its $(1-\alpha)4$ twice when on the inside similar to C plus extra $(1-\alpha)2$ when paired with C for total of $(1-\alpha)6$ plus extra $(1-\alpha)6$ unclaimed by {BC}.

We got the Shapley value for all α!

Define α procedure

Players join in random order

What happens when i joins a group that becomes S (with i included)?

Define marginal contributions S: For $j \in S$, $m_j(S) = v(S) - v(S \setminus j) \geq 0$.

Relabel players according to their marginal contribution: $m_1(S) \leq m_2(S) \leq \ldots \leq m_{|S|}(S)$

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When the player joining is $i < |S|$, the player gets αm_i

$i = |S|$, the player gets $\alpha m_{|S|} + (1 - \alpha)(m_{|S|} - m_{|S|-1|})$

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When the player joining is $i < |S|$, the player gets αm_i

$i = |S|$, the player gets $\alpha m_{|S|} + (1 - \alpha)(m_{|S|} - m_{|S|-1|})$

Each other player j in S gets $(1 - \alpha) \sum_{k=1}^{Min(i,j)} \frac{m_k - m_{k-1}}{|S| - k}$ where $m_0 = 0$

Innovation: Instead of dividing $(1-\alpha)m_i$ equally among all $|S|-1$ players, we divided it equally segment-by-segment among those who have a claim.

Theorem: Under the α procedure, average payoff to each player is its Shapley value

Proof: α procedure $i = 1$

What is average gain to player 1 in stage where $i \in S$ and set grows from $|S| - 1$ to $|S|$?

Two potential gains: (i) Go from $S \setminus 1$ to S and (ii) Go from $S \setminus j$ to S across all $j \neq 1$.

Player 1 gets $\frac{1}{|S|}[\alpha m_1(S) + \sum_{j \in S \setminus 1} (1 - \alpha) \sum_{k=1}^{min(1,j)} \frac{m_k - m_{k-1}}{|S| - k}]$ recall $m_0 = 0$

$$= \frac{1}{|S|}[\alpha m_1 + (|S| - 1)(1 - \alpha)\frac{m_1}{|S| - 1}] = \frac{m_1(S)}{|S|}$$

Payoff is independent of α. Case with $\alpha = 1$ is Shapley procedure. Hence Shapley value

What player 1 loses when joining is exactly made up from being inside

Proof: α procedure i = 2

What is average gain to player 2 in stage where $i \in S$ and set grows from $|S| - 1$ to $|S|$?

Two potential gains: (i) Go from $S \setminus 2$ to S and (ii) Go from $S \setminus j$ to S across all $j \neq 2$.

Player 2 gets $\frac{1}{|S|}[\alpha m_2(S) + \sum_{j \in S \setminus 2} (1 - \alpha) \sum_{k=1}^{min(2,j)} \frac{m_k - m_{k-1}}{|S| - k}]$

$$= \frac{1}{|S|}[\alpha m_2 + (1 - \alpha)[(|S| - 1)\frac{m_1}{|S| - 1} + (|S| - 2)\frac{m_2 - m_1}{|S| - 2}]]$$

$= \frac{m_2(S)}{|S|}$ is again independent of α. Case with $\alpha = 1$ is Shapley procedure.

Proof: α procedure i = |S|

What is average gain to player |S| in stage where $i \in S$ and set grows from $|S| - 1$ to $|S|$?

Two potential gains: (I) Go from $S \setminus |S|$ to S and (ii) Go from $S \setminus j$ to S across all $j \neq |S|$.

Player |S| gets $\dfrac{1}{|S|}[\alpha m_{|S|} + (1-\alpha)(m_{|S|} - m_{|S|-1}) + \sum_{j \in S,\, j<|S|} (1-\alpha) \sum_{k=1}^{min(|S|,j)} \dfrac{m_k - m_{k-1}}{|S| - k}]$

$$= \frac{1}{|S|}[\alpha m_{|S|} + (1-\alpha)(m_{|S|} - m_{|S|-1}) + (1-\alpha)[(|S|-1)\frac{m_1}{|S|-1} + \ldots + 1\frac{m_{|S|-1} - m_{|S|-2}}{1}]$$

$= \dfrac{m_{|S|}(S)}{|S|}$ is again independent of α. Case with $\alpha = 1$ is Shapley procedure.

To what extent is α procedure unique?

What happens when i joins a group that becomes S (with i included)?

As before, order the $|S|$ players

When the player joining is $i < |S|$, the player gets αm_i

$i = |S|$, the player gets $\alpha m_{|S|} + (1-\alpha)(m_{|S|} - m_{|S|-1|})$

How to divide up the remaining $(1-\alpha)m_i$? Or $(1-\alpha)m_{|S|-1}$ for $i = |S|$?

Consider more general weights $w_{j,k|i}(S) \geq 0$

Other players $j \neq i$ get $(1-\alpha)\sum_{k=1}^{i} w_{j,k|i}(m_k - m_{k-1})$ (stop at i since there is only m_i to distribute)

In our procedure, $w_{j,k|i}(S) = 1/(|S| - k)$ for $k \leq j$ and 0 for $k > j$

Our procedure uniquely leads to Shapley value subject to weights satisfying bargaining intuition

To what extent is α procedure unique?

Still true that $w_{i,k|j}(S) = 0$ for $k > i$.

Consider average gain to player i where $i \in S$ and set grows from $|S| - 1$ to $|S|$.

Two potential gains: (i) go from $S \setminus i$ to S, and (ii) go from $S \setminus j$ to S across all $j \neq i$.

For $i < |S|$, gain to player i is $\alpha m_i + \sum_{j \in S \setminus i} (1 - \alpha) \sum_{k=1}^{j} w_{i,k|j}(m_k - m_{k-1}) = m_i$ to get Shapley value

$$\sum_{j \in S \setminus i} \sum_{k=1}^{j} w_{i,k|j}(m_k - m_{k-1}) = m_i$$

Take $k = |S|$. Only term with $k = |S|$ comes from $j = |S|$.

Term $w_{i,|S|||S|}(m_{|S|} - m_{|S|-1})$ is only weight player i has on $m_{|S|}$. Therefore must be 0.

To what extent is α procedure unique?

$$\sum_{j \in S \setminus i} \sum_{k=1}^{j} w_{i,k|j}(m_k - m_{k-1}) = m_i$$

Take $k = |S| - 1$. Only terms with $k = |S| - 1$ come from $j = |S|$ and $j = |S| - 1$

The terms $w_{i,|S|-1||S|}(m_{|S|-1} - m_{|S|-2})$ and $w_{i,|S|-1||S|-1}(m_{|S|-1} - m_{|S|-2})$ are only weights on $m_{|S|-1}$

No negative weights. Therefore both must be 0 unless $i \geq |S| - 1$.

So yes: $w_{i,k|j}(S) = 0$ **for** $i < k$

$$\sum_{j \in S \setminus i} \sum_{k=1}^{Min(i,j)} w_{i,k|j}(m_k - m_{k-1}) = m_i$$

To what extent is α procedure unique?

$$\sum_{j \in S \setminus i} \sum_{k=1}^{Min(i,j)} w_{i,k|j}(m_k - m_{k-1}) = m_i$$

Let's look at positive weights. Take $k = |S| - 1$.

If $i = |S| - 1$ then only one term with $k = |S| - 1$: comes from $j = |S|$.

$w_{i,|S|-1||S|}(m_{|S|-1} - m_{|S|-2}) = m_{|S|-1}$ which implies $w_{i,|S|-1||S|} = 1$. Unique.

Also implies $w_{i,|S|-2||S|-2} + w_{i,|S|-2||S|} = 1$ to cancel out the -1. Not unique.

More generally, $w_{i,k|k+1} + w_{i,k|k+2} + \dots \ + w_{i,k||S|} = 1$ for $i \geq k$

and $\qquad w_{i,k+1|k} + w_{i,k+1|k+2} + \dots \ + w_{i,k+1||S|} = 1$ for $i \geq k+1$

To what extent is α procedure unique?

$$\sum_{j \in S \setminus i} \sum_{k=1}^{Min(i,j)} w_{i,k|j}(m_k - m_{k-1}) = m_i$$

Take $k = |S| - 2$

If $i = |S| - 2$ then two terms: $w_{i,|S|-2||S|}(m_{|S|-2} - m_{|S|-3})$ and $w_{i,|S|-2||S|-1}(m_{|S|-2} - m_{|S|-3})$.

In our procedure $w_{i,|S|-2||S|} = w_{i,|S|-2||S|-1} = 1/(|S| - k) = 1/2$.

Can now have $w_{i,|S|-2||S|} = \lambda$ and $w_{i,|S|-2||S|-1} = 1 - \lambda$.

Three players get a share of $m_{|S|-2}$. They are $|S| - 2, |S - 1|, |S|$. When on the inside, Player $i = |S| - 2$ gets λ when competing against player $|S| - 1$, and $1 - \lambda$ when against player $|S|$.

Bargaining Assumption

Share is larger (≥) when competing against weaker players: $w_{i,k|j_1} \geq w_{i,k|j_2}$ **if** $j_1 > j_2$

Bargaining assumption implies $\lambda \geq 1/2$. More generally …

Uniqueness Proof

Look at $w_{i,k|j}$ for general k. Only need consider $i \geq k$. By our bargaining assumption

$w_{k,k|k+1} \leq w_{k,k|k+2} \leq \ldots \leq w_{k,k||S|}$ therefore $w_{k,k|k+1} \leq 1/(|S| - k)$ and $w_{k,k||S|} \geq 1/(|S| - k)$

$w_{k+1,k|k} \leq w_{k+1,k|k+2} \leq \ldots \leq w_{k+1,k||S|}$ therefore $w_{k+1,k|k} \leq 1/(|S| - k)$ and $w_{k+1,k||S|} \geq 1/(|S| - k)$

$\ldots$

$w_{|S|-1,k|k} \leq w_{|S|-1,k|k+1} \leq \ldots \leq w_{|S|-1,k||S|}$ therefore $w_{|S|-1,k|k} \leq 1/(|S| - k)$ and $w_{|S|-1,k||S|} \geq 1/(|S| - k)$

$w_{|S|,k|k} \leq w_{|S|,k|k+1} \leq \ldots \leq w_{|S|,k||S|-1}$ therefore $w_{|S|,k|k} \leq 1/(|S| - k)$ and $w_{|S|,k||S|-1} \geq 1/(|S| - k)$

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$w_{k,k||S|} + w_{k+1,k||S|} + \ldots + w_{|S|-1,k||S|} = 1$

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$\ldots$

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$w_{k,k||S|} + w_{k+1,k||S|} + \ldots + w_{|S|-1,k||S|} = 1$

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$\ldots$

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$w_{k,k||S|} + w_{k+1,k||S|} + \ldots + w_{|S|-1,k||S|} = 1$

Therefore $w_{k,k||S|} = w_{k+1,k||S|} = \ldots = w_{|S|-1,k||S|} = 1/(|S|-k)$

Therefore $\boxed{w_{i,k|j} = 1/(|S|-k) \text{ for } k \leq i < |S|, k \leq j \leq |S|, i \neq j.}$

Uniqueness Proof

Look at $w_{i,k|j}$ for general k. Only need consider $i \geq k$. By our bargaining assumption

$w_{k,k|k+1} \leq w_{k,k|k+2} \leq \ldots \leq w_{k,k||S|}$ therefore $w_{k,k|k+1} \leq 1/(|S|-k)$ and $w_{k,k||S|} \geq 1/(|S|-k)$

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$\ldots$

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$w_{|S|,k|k} \leq w_{|S|,k|k+1} \leq \ldots \leq w_{|S|,k||S|-1}$ therefore $w_{|S|,k|k} \leq 1/(|S|-k)$ and $w_{|S|,k||S|-1} \geq 1/(|S|-k)$

$w_{k,k||S|} + w_{k+1,k||S|} + \ldots + w_{|S|-1,k||S|} = 1$

Therefore $w_{k,k||S|} = w_{k+1,k||S|} = \ldots = w_{|S|-1,k||S|} = 1/(|S|-k)$

Therefore $w_{i,k|j} = 1/(|S|-k)$ for $k \leq i < |S|$, $k \leq j \leq |S|$, $i \neq j$.

But $w_{k+1,k|k} + w_{k+2,k|k} + \ldots + w_{|S|,k|k} = 1$ and therefore $w_{|S|,k|k} = 1/(|S|-k)$

Uniqueness Proof

Look at $w_{i,k|j}$ for general k. Only need consider $i \geq k$. By our bargaining assumption

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$w_{|S|,k|k} \boxed{\leq w_{|S|,k|k+1} \leq \ldots \leq w_{|S|,k||S|-1}}$ therefore $w_{|S|,k|k} \leq 1/(|S| - k)$ and $w_{|S|,k||S|-1} \geq 1/(|S| - k)$

$w_{k,k||S|} + w_{k+1,k||S|} + \ldots + w_{|S|-1,k||S|} = 1$

Therefore $w_{k,k||S|} = w_{k+1,k||S|} = \ldots = w_{|S|-1,k||S|} = 1/(|S| - k)$

Therefore $w_{i,k|j} = 1/(|S| - k)$ for $k \leq i < |S|, k \leq j \leq |S|, i \neq j$.

But $w_{k+1,k|k} + w_{k+2,k|k} + \ldots + w_{|S|,k|k} = 1$ and therefore $w_{|S|,k|k} = 1/(|S| - k)$

This implies $\boxed{w_{|S|,k|k} = w_{|S|,k|k+1} = \ldots = w_{|S|,k||S|-1} = 1/(|S| - k).}$

Uniqueness of α Procedure

Bargaining Assumption:

Share is larger (not smaller) when competing against weaker players.

If weights lead to Shapley value

$$\sum_{j \in S \setminus i} \sum_{k=1}^{j} w_{i,k|j}(m_k - m_{k-1}) = m_i$$

then $w_{j,k|i}(S) = 1/(|S| - k)$ for $k \leq j$ and 0 for $k > j$

That's our procedure

Running α Procedure in Reverse

Procedure has people join one at a time in random order.

Can also start with grand coalition and assume one person is at risk of being eliminated if they don't reach an agreement.

Shapley-like procedure says person i at risk just gets full marginal contribution while other players get their payoff in the procedure applied to $N \backslash i$.

Need to apply backward induction to solve.

This is foundation of non-cooperative game in Hart and Mas-Colell (1996).

Running α Procedure in Reverse

Procedure has people join one at a time in random order.

Can also start with grand coalition and assume one person is at risk of being eliminated if they don't reach an agreement

Our procedure says person i at risk negotiates a split of $(\alpha, 1 - \alpha)$ of marginal contribution plus gives other players their payoff in the procedure applied to $N \backslash i$

Need to apply backward induction to solve.

If everyone has equal chance of being at risk, expected payoffs are independent of α and result is Shapley. All (reverse) orderings are equally likely. Reverse α procedure yields same answer.

Potential Generalization

If no agreement, who should be eliminated?

Chance of being eliminated could depend on marginal contribution.

Payoffs then depend on α.

In airline case, A (who only uses runway of length 1) would be most likely to be eliminated.

Procedures and non-cooperative games

Not a set of axioms

Not a non-cooperative game

In-between a cooperative game with axioms and a non-cooperative game rules

Interesting in their own right

Procedure motivates non-cooperative game

Suggests different approach to weighted Shapley value, one where weights are endogenous

Non-Cooperative Game

Literature Review

Hart and Mas-Colell (1996)

One person is chosen at random to make offer to others. If anyone rejects, person is excluded and gets 0

If person at risk is person making the offer, can ask for full marginal contribution. Same as Shapley value, except process runs in reverse

If chance of not being kicked out is ρ, not 0, expected value of payoffs remains unchanged. But all payoffs converge to Shapley value as $\rho \to 1$

In generalization, person at risk of exclusion may be different from person making offer. Similar to $(\alpha, 1-\alpha)$ model, but $(1-\alpha)$ division is always equal or weighted by player identity.

TU to NTU solution as in Maschler and Owen (1992)

Gul (1989, 1999)

People meet in groups. Person joining sells their right to negotiate to group leader. If equilibrium is efficient, then must equal Shapley value.

Non-Cooperative Game: Ground rules

As before: Define marginal contributions S: For $j \in S$, $m_j(S) = v(S) - v(S \setminus j) \geq 0$.

Relabel players according to their marginal contribution: $m_1(S) \leq m_2(S) \leq \ldots \leq m_{|S|}(S)$

We adopt Hart and Mas-Colell rules for accepting offers. Everyone is asked in turn if they accept. Accept if indifferent. If any reject, person at risk is excluded with chance $1 - \rho$.

We focus on case where no agreement leads to exclusion with probability 1. ($\rho = 0$)

Who gets to make offers? Intuition for the game's rules is similar to intuition for procedure.

Non-Cooperative Game: Ground rules cont.

Everyone knows who is at risk when making the proposal and accepting/rejecting it

(Also fine if player making proposal only has probability distribution on who is at risk.)

$\tau_{j,i}(S, v_S)$ is chance player j makes proposal and i is at risk

$\tau_{j|i}(S, v_S)$ is chance player j makes proposal given i is at risk

where v_S is the characteristic function restricted to subsets of S. In particular, $\tau_{j|i}$ can depend on marginal contributions

For ease of notation, we write $\tau_{j|i}(S)$.

Hart and Mas-Colell Theorem (adapted for $\rho = 0$)

$\phi^k(S)$ is player k's equilibrium expected payoff in stationary subgame perfect equilibrium

$\beta_k(S)$ is player k's chance of being the one excluded if no deal

Theorem: In the unique stationary subgame-perfect equilibrium:

$$\phi^k(S) = \sum_{i \in S} \beta_i(S)[\phi^k(S \setminus i) + \tau_{k|i}(S) m_i(S)]$$

Note that $\phi^k(S \setminus k) = 0$

Say i is at risk. If someone other than k makes the offer, k gets $\phi^k(S \setminus i)$. If k makes the offer, k gets $\phi^k(S \setminus i) + m_k(S)$.

Player making offer will ask for (and get) marginal contribution of player at risk of exclusion and give everyone (self included) payoff in game without that player

Corollary

Theorem: In the unique stationary subgame-perfect equilibrium:

$$\phi^k(S) = \sum_{i \in S} \beta_i(S)[\phi^k(S \setminus i) + \tau_{k|i}(S) m_i(S)]$$

Payoffs coincide with Shapley value if and only if

$$\phi^k(S) = \frac{1}{|S|}[m_k(S) + \sum_{i \in S} \phi^k(S \setminus i)]$$

Corollary (Hart-Mas-Colell): Stationary subgame-perfect equilibrium coincides with Shapley value for all TU games if and only if: $\beta_k(S) = 1/|S|$, $\tau_{k|k}(S) = 1$, $\tau_{k|i}(S) = 0$ when $i \neq k$, for all $k \in S \subseteq N$.

The IF part is clear. That leads to Shapley recursion formula.

ONLY IF: To obtain the Shapley value, only the marginal contribution of k can matter (Young, 1985); therefore $\tau_{k|i}(S) = 0$ when $i \neq k$, and $\tau_{k|k}(S) = 1$.

Corollary

"Corollary says that to obtain the Shapley value one needs, first, that only proposers (but not responders) may drop out; and second, that the probabilities [of] dropping out should be equalized across the players. The first condition is related to the null player axiom, and the second to the symmetry axiom." —Hart and Mas-Colell (1996)

New probabilities

Player i is at risk of being eliminated if no agreement.

Player j's chance of making the proposal given i is at risk is $\tau_{j|i}$ as defined below

With chance α, player i is chosen to make a proposal to the other players.

With chance $1 - \alpha$, the not-at-risk players are given priority in terms of making a proposal. A proposer is chosen according to the following procedure

1. A number is selected at random from the interval $[0, m_i(S)]$.
2. Any player j whose marginal contribution $m_j(S)$ is above the number selected is in the eligible pool;
3. All players in the eligible pool are selected with equal probability;
4. If no player is in the eligible pool, player i makes the proposal.

Someone always makes the proposal. Thus $\sum_j \tau_{j|i} = 1$ for all i.

New probabilities

Player j's chance of making the proposal given i is at risk is

$$\tau_{i|i} = \alpha \text{ for } j = i < |S|,$$

$$\tau_{i|i} = \alpha + \frac{(1-\alpha)}{m_{|S|}}[m_{|S|} - m_{|S|-1}] \text{ for } j = i = |S|,$$

$$\tau_{j|i} = \frac{(1-\alpha)}{m_i(S)} \sum_{k=1}^{Min(i,j)} \frac{m_k - m_{k-1}}{|S| - k} \quad \text{for } j \neq i$$

Game mimics procedure. Person i at risk has some power and thus gets $\alpha m_i(S)$, and other players split up the remaining $(1-\alpha)m_i(S)$

With $0 \leq \alpha < 1$, the $\tau_{j|i}$ are more general than Hart & Mas-Colell since they depend on the game

$\alpha = 1$

Leads to Shapley via Hart and Mas-Colell: $\tau_{i|i} = 1$ and $\tau_{j|i} = 0$ for $j \neq i$.

Player making proposal gets marginal contribution.

We pick person to make offer conditional on knowing who will be excluded.
Hart and Mas-Colell pick exclusion person conditional on knowing who will make offer.
Different game trees. With $\alpha = 1$, effectively the same.

$$\tau_{j|i} = \alpha \text{ for } j = i < |S|,$$

$$\tau_{j|i} = \alpha + \frac{(1-\alpha)}{m_{|S|}}[m_{|S|} - m_{|S|-1}] \text{ for } j = i = |S|,$$

$$\tau_{j|i} = \frac{(1-\alpha)}{m_i(S)} \sum_{k=1}^{Min(i,j)} \frac{m_k - m_{k-1}}{|S| - k} \quad \text{for } j \neq i$$

General α

Theorem: In any stationary subgame-perfect equilibrium:

$$\phi^k(S) = \sum_{i \in S} \beta_i(S)[\phi^k(S \setminus i) + \tau_{k|i}(S) m_i(S)]$$

Corollary: If $\beta_k(S) = 1/|S|$ and $\tau_{k|i}$ is as defined, then

$$\phi^k(S) = \frac{1}{|S|}[m_k(S) + \sum_{i \in S} \phi^k(S \setminus i)]$$

$\phi^k(S)$ is not a function of α. Hence same as $\alpha = 1$. Hence $\phi^k(S) = Sh^k(S)$.

Shapley recursion formula. Hence $\phi^k(S) = Sh^k(S)$.

Key Step

$$\phi^k(S) = \frac{1}{|S|} \sum_{i \in S} [\phi^k(S \setminus i) + \tau_{k|i}(S) m_i(S)]$$

Result depends on

$$\sum_{i \in S} \tau_{k|i}(S) m_i(S) = m_k(S)$$

Player k's expected payoff from making offers depends only on its marginal contribution.

Proof (for $j < |S|$)

$$\tau_{j|i}(S)m_i(S) = (1-\alpha)\sum_{k=1}^{Min(i,j)} \frac{m_k - m_{k-1}}{|S|-k} \quad \text{for } j \neq i$$ which is completely symmetric between i and j

$$\implies \sum_{i\in S,\, i\neq j} \tau_{j|i}(S)m_i(S) = \sum_{i\in S,\, i\neq j} \tau_{i|j}(S)m_j(S) = m_j(S) \sum_{i\in S,\, i\neq j} \tau_{i|j}(S)$$

For $j < |S|$, $\tau_{j|j}(S) = \alpha$ and $\sum_{i\in S,\, i\neq j} \tau_{i|j}(S) = 1-\alpha$

$$\sum_{i\in S} \tau_{j|i}(S)m_i(S) = \alpha m_j(S) + (1-\alpha)m_j(S) = m_j(S)$$

Proof (for $j = |S|$)

$$\tau_{j|i}(S)m_i(S) = (1-\alpha)\sum_{k=1}^{Min(i,j)} \frac{m_k - m_{k-1}}{|S| - k} \quad \text{for } j \neq i$$ which is completely symmetric between i and j

$$\implies \sum_{i\in S,\, i\neq j} \tau_{j|i}(S)m_i(S) = \sum_{i\in S,\, i\neq j} \tau_{i|j}(S)m_j(S) = m_j(S) \sum_{i\in S,\, i\neq j} \tau_{i|j}(S)$$

For $j = |S|$, $\tau_{j|j}(S) = \alpha + (1-\alpha)\frac{m_{|S|} - m_{|S|-1}}{m_{|S|}}$ and $\sum_{i\in S,\, i\neq j} \tau_{i|j}(S) = (1-\alpha)\frac{m_{|S|-1}}{m_{|S|}}$

$$\sum_{i\in S} \tau_{j|i}(S)m_i(S) = \alpha m_{|S|} + (1-\alpha)(m_{|S|} - m_{|S|-1}) + (1-\alpha)m_{|S|-1} = m_{|S|}(S)$$

Corollary

Theorem: In the unique stationary subgame-perfect equilibrium:

$$\phi^k(S) = \sum_{i \in S} \beta_i(S)[\phi^k(S \setminus i) + \tau_{k|i}(S) m_i(S)]$$

Corollary (Hart-Mas-Colell): Stationary subgame-perfect equilibrium coincides with Shapley value for all TU games if and only if: $\beta_k(S) = 1/|S|$, $\tau_{k|k}(S) = 1$, $\tau_{k|i}(S) = 0$ when $i \neq k$, for all $k \in S \subseteq N$.

ONLY IF: To obtain the Shapley value, only the marginal contribution of k can matter; therefore $\tau_{k|i}(S) = 0$ when $i \neq k$ if $\tau_{k|i}(S)$ must be the same for *all* games.

If $\tau_{k|i}(S)$ can depend on the game (vary w. marginal contributions) then

ONLY IF: $\sum_{i \in S} \tau_{k|i}(S) m_i(S) = m_k(S)$

Person at risk need not be same person as one making proposal.

Payoffs in non-cooperative game same as procedure

Player j's chance of making the proposal given i is at risk is

$$\tau_{j|i} = \alpha \text{ for } j = i < |S|,$$

$$\tau_{j|i} = \alpha + \frac{(1-\alpha)}{m_{|S|}}[m_{|S|} - m_{|S|-1}] \text{ for } j = i = |S|,$$

$$\tau_{j|i} = \frac{(1-\alpha)}{m_i(S)} \sum_{k=1}^{Min(i,j)} \frac{m_k - m_{k-1}}{|S| - k} \quad \text{for } j \neq i$$

Exactly the same as player j's share of $m_i(S)$ when i joins S in procedure. Same uniqueness result.

Alternative game: Don't know who will be excluded

Will only discover if agreement fails.

Look at case where everyone has equal chance of being excluded no matter who makes the offer: $\tau_{i|j} = 1/|S|$ and therefore $\beta_i(S) = \sum_j \sigma_j(S)\tau_{i|j} = 1/|S| \sum_j \sigma_j(S) = 1/|S|$

Chance of making an offer can depend on marginal contribution: $\sigma_i = \frac{m_i}{\sum_{j \in s} m_j}$. Seems intuitive.

Person making offer will ask for average value of marginal contribution: $\sum_{k \in S} \tau_{k|i} m_k(S) = \frac{1}{|S|} \sum_k m_k$

Expected gain to i is equal to marginal contribution: $\frac{m_i}{\sum_j m_j} \times \frac{1}{|S|} \sum_j m_j = \frac{m_i}{|S|}$

 Result will be the Shapley value.

If chance of making offer and chance of exclusion are equal to $1/|S|$ for all players

Then only $\tau_{i|j}$ that leads to Shapley value has person making the offer being at risk of exclusion if rejected.

Consider player 1, the player with lowest marginal contribution.

Smallest payoff that can be assigned is when player 1 is the one at risk of exclusion when making the offer. Expected payoff is $m_1(S)/|S|$. Shapley payoff.

Have now used up all $1/|S|$ exclusion probability for player 1.

Thus minimum possible payoff to player 2 is when all $1/|S|$ exclusion probability for player 2 is assigned to player 2. Expected payoff is $m_2(S)/|S|$. Shapley payoff.

If chance of exclusion is equal to $1/|S|$ for all players

Either player at risk of exclusion also makes the offer

Or, if players other than excluded player can make the offer, can pick intuitive $\tau_{j|i}$ that lead to Shapley.

We have seen two examples that vary based on information structure.

But chance of making offers will differ across players.

Other offer rules from Hart and Mas-Colell

Chance i is at risk given that j makes proposal is

(i) $\tau_{i|i} = 0$, $\tau_{i|j} = 1/(|S| - 1)$ for $i \neq j$.

Think of $(\alpha, 1 - \alpha)$ with $(1 - \alpha)$ evenly divided across $|S| - 1$ and $\alpha = 0$. This leads to all players, including dummy players, getting $v(N)/|N|$.

(ii) $\tau_{i|j} = 1/|S|$ for all (i, j)

Think of $(\alpha, 1 - \alpha)$ with $(1 - \alpha)$ evenly divided across $|S| - 1$ and $\alpha = 1/|S|$. Not quite equal payoffs to all, but close as N gets large

(iii) Gomes (1991) $\sigma_i(S) = \dfrac{w_i}{\sum_{j \in s} w_j}$ and $\tau_{i|i} = 1$ (only proposer drops out)

Result is weighted Shapley value. Note that weights w_i don't vary with S

A non-cooperative game with endogenous weights

Stick with $\tau_{i|i} = 1$ so that only proposer is at risk: Thus $\beta_i(S) = \sigma_i(S)$.

Want weights w_i to be connected to marginal contributions. Lower marginal contribution should lead to higher chance of exclusion.

If some $m_i = 0$, these players are the only ones at risk of exclusion (and probabilities are equal).

If $m_i > 0$ for all i, $w_i = m_i(S)^\lambda$ so that $\beta_i(S) = \sigma_i(S) = \dfrac{m_i^\lambda}{\sum_{j \in S} m_j^\lambda}$.

Note weights are always equal in two-person games (since m_i's are equal). Hence not weighted Shapley value. Outcomes now DO depend on α.

We focus on $\lambda \leq 0$. At $\lambda = 0$, weights are equal and we have Shapley value. As $\lambda \to -\infty$, player(s) with lowest marginal contribution in S will be excluded.

Endogenous weights example: Runway revisited

Airline A needs runway of length 1. Marginal contribution is 1.
(B needs length 2, C needs length 3.)

$\alpha = 0.5$: Under $\lambda = 0$, airline A pays 1/3, gains 2/3 (Shapley value)

Under $\lambda \to -\infty$, airline A pays 1/2 (nucleolus). Airline A is negotiating with {BC} pair.

B and C split cost of second leg (in expectation), airline C always pays full cost of third leg.

$\alpha = 1$: Airline A pays 1/3 at $\lambda = 0$, 1/4 at $\lambda = -1$, and 0 as $\lambda \to -\infty$.

Whether being excluded is good or bad for a player depends on whether αm_i is above or below its Shapley value.

Interim summary

The players are all negotiating with each other.

If they don't reach a deal then in non-cooperative game, one would get picked to be at risk of exclusion and someone else will get to make offer.

This determines what negotiation researchers call BATNAs or economists call subgame payoffs

If ABC can't reach a deal, what will happen?

A two-way deal? Are all three two-way deals equally likely?

Non-cooperative game and procedure both determine the result of no-agreement.

Two things: (1) how to split up gains when someone makes a proposal (α, and split of $1 - \alpha$); (2) how to determine who is at risk.

NTU games

Define "α marginal contribution" of i to S as follows. Take ordering π.

Start with payoffs to each player 1, 2, … j.

Player j+1 in the ordering gets α while the other players get

Not really possible to define all the marginal orderings at the same time. Could define it as the average d_i over all orderings without that player.

NTU games

Maschler and Owen (1989, 1992) define "marginal contribution" of i to S as follows. Take ordering π.

Player 1 gets V(1).

Player 2 gets max payoff possible subject to 1 getting V(1).

And so forth

This approach does not define all the marginal orderings at the same time.

Can define player i's average marginal contribution to a set S (by looking at all possible orderings), but not Player i's marginal contribution.

NTU games

We define “marginal contribution” of i to S as follows. Note that marginal contribution depends on solution concept and is defined by induction.

Consider solution to $S \setminus i$. Give all players other than i those payoffs. Then find max payoff that can be given to i so that the profile is still feasible (is in $V(S)$).

Can define $d^j(S)$ in this manner for all players j in S. Relabel the players in S so that $d^j(S)$ is increasing.

Solution concept: In going from $S \setminus i$ to S we move in direction z until we hit the frontier, where for $i < |S|$, $z_i = \alpha$ and for $j \neq i$

$$z_j = \frac{1-\alpha}{d_i(S)} \sum_{k=1}^{Min(i,j)} \frac{d_k(S) - d_{k-1}(S)}{|S| - k}.$$

We can think of this as being motivated by the game where player i is the player at risk of being eliminated and the different components of the vector represent the probabilities of who gets to make the proposal multiplied by the associated gains. The case of $\alpha = 1$ is the standard case.

If feasible sets are all hyperplanes, then we can give each player their expected value and that will be feasible. That is the inductively defined solution to S when feasible set is a hyperplane.

NTU games

Look at all hyperplanes to attainable set. Find solution assuming hyperplane is feasible. Then find fixed point as in Nash. That is general solution to S.

Maschler and Owen (1989, 1992) show (given some regularity conditions) there is always a fixed point where hyperplane solution is feasible. May not be unique.

In Hart and Mas-Colell move to frontier by giving everything to A. If A makes offer, gets to keep all surplus subject to giving {BC} their expected value under game without A (which is also a hyperplane game).

We do something similar using $(\alpha, 1-\alpha)$ split rather than $(1,0)$ split.

Consistency is similar to Nash IIA argument. *Can find other consistent solutions using our generalized Shapley value to hyperplane games.*

In the case of pure bargaining games, $\alpha = 1$ leads to Nash bargaining solution.

$z = \frac{1}{|S|}\mathbf{1}$ leads to Kalai-Smorodinsky solution.

Summary

Move from $(1,0)$ to $(\alpha,1-\alpha)$ in negotiation procedure / game.

More realistic (and more complicated) procedure still leads to Shapley value. Surprising.

Helps parties understand and accept Shapley value (and also understand its implicit assumptions).

In non-cooperative game, don't need person making offer to also be person at risk of exclusion. More realistic (and more complicated) game still leads to Shapley value if chance of being excluded is equal.

If risk of exclusion is not equal, outcome depends on α.

Suggests new endogenous approach to weighting in Shapley value.

Suggests new way to move from TU to NTU solutions.